Design of State Feedback Observer based on Kharitnov Polynomial for Rotary Inverted Pendulum

Jim George* and Dr. Gylson Thomas**
* Assistant Professor, Department of EEE, Muthoot Institute of Technology and Science, Ernakulam, Kerala
jimgeorge@mgits.ac.in
** Professor, Department of EEE, Muthoot Institute of Technology and Science, Ernakulam, Kerala
gylsonthomas@mgits.ac.in

Abstract: Rotary Inverted Pendulum is a nonlinear, dynamic unstable system. The key idea is to design a state feedback observer to stabilize the position of pendulum in inverted position which is mounted on a rotary base which is the stabilization point of the system. This paper deals with mathematical modeling, design of full order state feedback controller and case study with pole placement based design.

Keywords: Rotary Inverted Pendulum(RIP), Kharitnov Polynomial, State feedback Observer.

Introduction

Rotary Inverted Pendulum(RIP) consists of a pendulum attached to a rotating arm driven by a motor. So RIP is a two degree of freedom (DOF) system. The two DOFs are angle of rotation of the rotating arm and angle of rotation of the pendulum. The rotating arm acts as the base for the inverted pendulum which means that the pendulum is coupled to a rotary base. For this system a full order state observer needs to be designed. The aim is to design a full order state observer gain matrix such that the range of each gain value is determined by Kharitnov polynomial technique. The simulation results show the stability of the system and error analysis for the designed observer for various gain values.

Rotary Inverted Pendulum

Fig. 1 Rotary Inverted Pendulum

A RIP consists of dc servomotor, rotating arm and pendulum attached via encoder to the rotating arm. Rotating arm is attached to the rotor shaft of dc servomotor through a gear system.

System Modeling

System modeling consists of two parts i)Modeling of dc servomotor ii) Modeling of rotating arm and pendulum system.

Modeling of dc servomotor

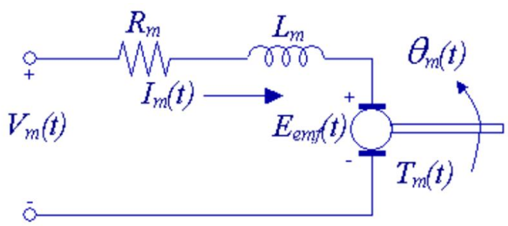

Fig 2: Equivalent model of armature controlled dc servomotor

R_m is the resistance of the armature circuit

L_m is the inductance of the armature circuit

E_{emf} is the back emf developed

θ_m (q_3) is the angular displacement of the motor shaft or angular displacement of the rotating arm

T_m is the torque developed

By Kirchoff's voltage law:

$$V_m - R_m I_m - L_m \dot{I_m} - E_{emf} = 0 \tag{1}$$

By torque current relationship,

$$T \alpha I_m \tag{2}$$

The above proportionality can be written as:

$$T = K_t I_m \tag{3}$$

Where K_t is the torque constant.

The torque equation can be written as:

$$T = J\dot{\omega} + B\omega \tag{4}$$

Where

J is moment of inertia

B is rotational frictional coefficient

The expression for back emf can be written as:

$$E_{emf} = K_b \omega \tag{5}$$

Where

K_b is the back emf constant

ω is the angular velocity

$$\omega = \dot{\theta} \tag{6}$$

From the above equations, the transfer function of the dc servomotor is obtained as:

$$\frac{\theta(s)}{V_m(s)} = \frac{K_t K_b}{s(s^2 J L_m + s(B L_m + J R_m) + R_m B + K_t K_b)} \tag{7}$$

Taking into account, efficiencies of gear box and motor,

The transfer function can be written as:

$$\frac{\theta(s)}{V_m(s)} = \frac{\eta_m \eta_g K_t K_b}{s(s^2 J L_m + s(B L_m + J R_m) + R_m B + \eta_m \eta_g K_t K_b)} \tag{8}$$

Where:

$$K_b = K_m K_g \tag{9}$$

Since

$$L_m \ll R_m$$

$$\frac{\theta(s)}{V_m(s)} = \frac{\eta_m \eta_g K_t K_b}{s^2 (J R_m) + s(R_m B + \eta_m \eta_g K_t K_b)} \tag{10}$$

Where,

η_m is the efficiency of the motor

η_g is the efficiency of the gear box

Modeling of rotating arm and pendulum

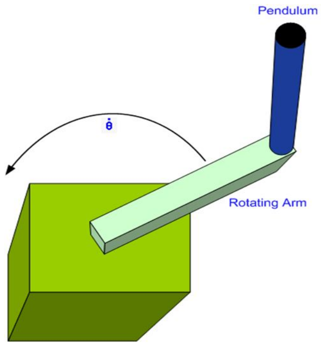

Fig 3: Top view of rotary inverted pendulum

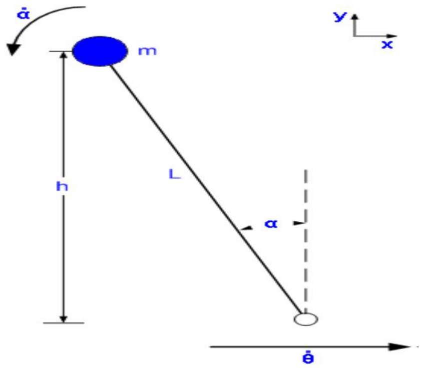

Fig 4: Side view of the pendulum in motion

L is the length to pendulum's centre of mass (half the pendulum length)

m is the mass of the pendulum

r is the half the length rotating arm

α is the pendulum deflection

θ is the servo load gear angle

From fig 4, there are two components for velocity of the pendulum lumped mass:

$$V_{pendlumum\ centre\ of\ mass} = -Lcos\alpha(\dot{\alpha})\hat{x} - Lsin\alpha(\dot{\alpha})\hat{y} \tag{11}$$

Velocity of the rotating arm:

$$V_{arm} = r\dot{\theta} \quad (12)$$

Taking the x and y components:

$$V_x = r\theta - L\cos\alpha(\dot{\alpha}) \quad , \quad (13)$$

$$V_y = -L\sin\alpha(\dot{\alpha}) \quad (14)$$

System Dynamic Equations

Potential Energy is only due to gravity.

$$PE_{pendulum} = mgh = mgL\cos\alpha \quad (15)$$

Kinetic Energy arises from the moving hub, the velocity of point mass in the x-direction, the velocity of the point mass in the y-direction and the rotating pendulum about its center of mass:

$$T = KE_{Hub} + KE_{Vx} + KE_{Vy} + KE_{pendulum} \quad (16)$$

The moment of inertia of a rod about its centre of mass is:

$$J_{cm} = \frac{1}{12}MR^2 \quad (17)$$

Taking: $R = 2L$

$$J_{cm} = \frac{1}{3}ML^2$$

$$KE_{Hub} = \frac{1}{2}J\dot{\theta}^2$$

$$KE_{Vx} = \frac{1}{2}mV_x^2$$

$$KE_{Vy} = \frac{1}{2}mV_y^2$$

$$KE_{pendulum} = \frac{1}{2}J_{cm}\dot{\alpha}^2$$

The Lagrangian is given by:

$$L = T - PE_{pendulum} \quad (18)$$

$$\frac{\delta}{\delta t}\left(\frac{\delta L}{\delta \dot{\theta}}\right) - \frac{\delta L}{\delta \theta} = T_{output} - B_{eq}\dot{\theta} \quad (19)$$

$$\frac{\delta}{\delta t}\left(\frac{\delta L}{\delta \dot{\alpha}}\right) - \frac{\delta L}{\delta \alpha} = 0 \quad (20)$$

Output torque on the load of the motor is:

$$T_{output} = \frac{\eta_m \eta_g K_t K_g (V_m - K_g K_m \theta)}{R_m} \quad (21)$$

State space model of the system

The state space model of one sided rotary inverted pendulum can be deduced from the above expressions as shown below,

$$\begin{bmatrix} \dot{\theta} \\ \dot{\alpha} \\ \ddot{\theta} \\ \ddot{\alpha} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{bd}{E} & \frac{-cG}{E} & 0 \\ 0 & \frac{ad}{E} & \frac{-bG}{E} & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \alpha \\ \dot{\theta} \\ \dot{\alpha} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{c\eta_m\eta_g K_t K_g}{R_m E} \\ \frac{b\eta_m\eta_g K_t K_g}{R_m E} \end{bmatrix} V_m \quad (22)$$

Where:

$a = J + mr^2$

$b = mLr$

$c = \frac{4}{3}mL^2$

$d = mgL$

$E = ac - b^2$

$$G = \frac{\eta_m \eta_g K_t K_m K_g^2 + BR_m}{R_m}$$

State Model Of Rip System

Table 1: Specifications of Pendulum

Specificatioms	Pendulum
Mass (m)	.2kg
Length (L)	.3m
Motor constant (K_t)	.00767
Gear ratio (K_g)	70
Rotational frictional coefficient (B)	.004
Efficiency of gear (η_g)	.9
Efficiency of motor (η_m)	.69
Armature resistance (R_m)	2.6ohm

Transfer function of the system

From the above specifications, transfer function of the system is given by:

$$\frac{\alpha(s)}{V_m(s)} = \frac{16.02s + .06292}{s^3 + 18.21s^2 - 61.31s - 446.9}$$

Observability

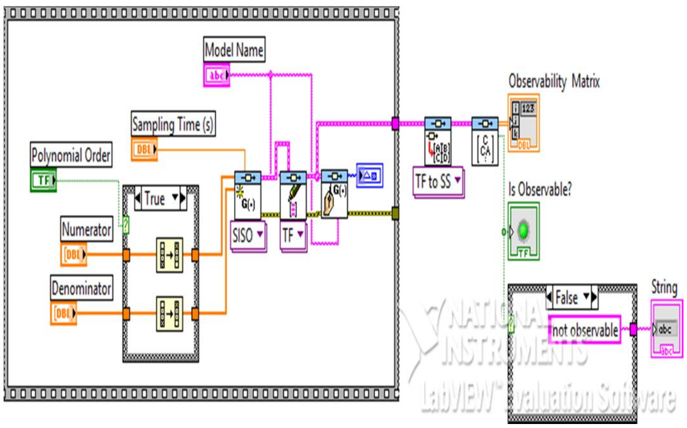

Fig 5: Block diagram to check observability

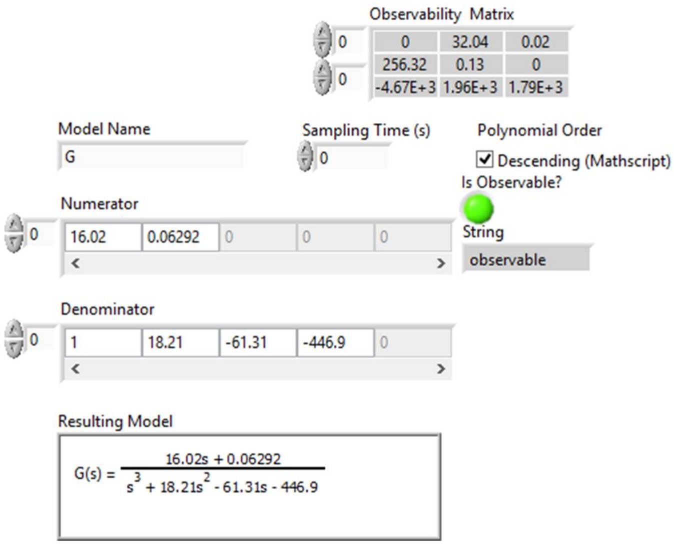

Fig 6: Front panel to check observability

The observability analysis is performed using LabVIEW and the system is completely state observable. Hence a full order state observer canbe designed for the system.

State Observer

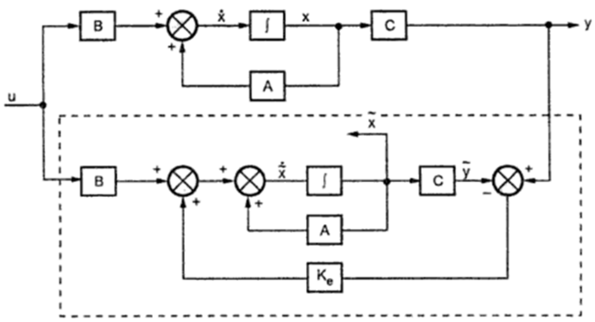

Fig 7: Full order state observer

The state space model of RIP system is given as

$$\mathbf{dx/dt} = \begin{bmatrix} -18.21 & 7.66375 & 6.98281 \\ 8 & 0 & 0 \\ 0 & 8 & 0 \end{bmatrix} \mathbf{x(t)} + \begin{bmatrix} 0.0625 \\ 0 \\ 0 \end{bmatrix} \mathbf{u(t)}$$

$$\mathbf{y(t)} = \begin{bmatrix} 0 & 32.04 & 0.01573 \end{bmatrix} \mathbf{x(t)} + \begin{bmatrix} 0 \end{bmatrix} \mathbf{u(t)} \tag{23}$$

This in the form

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

The state equation of Full order state observer is given by

$$\dot{\tilde{x}} = A\tilde{x} + Bu + K_e(y - C\tilde{x}) \qquad (24)$$

The error equation of the observer is

$$\begin{aligned} \dot{x} - \dot{\tilde{x}} &= [Ax + Bu] - [A\tilde{x} + Bu + K_e(y - C\tilde{x})] \\ &= Ax - A\tilde{x} - K_e(y - C\tilde{x}) \\ &= A(x - \tilde{x}) - K_e(Cx - C\tilde{x}) \\ &= A(x - \tilde{x}) - K_e\, C(x - \tilde{x}) \\ \dot{x} - \dot{\tilde{x}} &= (A - K_eC)\,(x - \tilde{x}) \end{aligned} \qquad (25)$$

$$\begin{aligned} e &= x - \tilde{x} \\ \dot{e} &= (A - K_eC)e \end{aligned} \qquad (26)$$

The dynamic behavior of the system is obtained from eigen values of the matrix given in (26). If [A-K_eC] is a stable matrix then error will converge to zero. So to obtain range of K_e so that eigen values lies on stable region, Kharitnov polynomial technique is used.

Kharitnov Polynomial

A theory concerned with the root locations for a family of polynomials (cf. Polynomial). A good general reference for this area is . The motivation for this theory derives from the issue of robust stability for systems of linear time-invariant differential equations. For a system of linear differential equations (cf. Differential equation, ordinary) $\dot{x} = Ax$

Stability is determined by the roots of the characteristic polynomial

$$p(s) = a_0 + a_1 s^2 + \cdots + a_n s^n = \det(sI - A + K_eC) \qquad (27)$$

The system of differential equations will be stable if and only if all roots of this characteristic polynomial lie in the open left half of the complex plane. In this case, the polynomial is said to be Hurwitz stable. For a single polynomial, the question of stability can be determined using the Routh–Hurwitz criterion.

The question of robust stability arises when it is supposed that the system of differential equations depends on uncertain parameters whose values are unknown but satisfy known bounds. The presence of such uncertain parameters means that the coefficients of the characteristic polynomial are unknown but bounded. This then defines a family of characteristic polynomials. The system will be robustly stable if all polynomials in this family have all their roots in the open left half of the complex plane.

The most important result in this area is a theorem due to V.L. Kharitonov . In this result, the polynomial family considered is a collection of polynomials with the following specific form:

$$p(s) = a_0 + a_1 s^2 + \cdots + a_n s^n \qquad (3.32)$$

$$a_i^- \le a_i \le a_i^+, i = 0, \ldots\ldots, n \qquad (3.33)$$

Thus, each coefficient of the polynomial is contained within a given interval. Such a polynomial family is referred to as an interval polynomial. Kharitonov's theorem gives a necessary and sufficient condition for the robust stability of such an interval polynomial in terms of the following four polynomials:

$$p_1(s) = a_0^- + a_1^- s + a_2^+ s^2 + a_3^+ s^3 + a_4^- s^4 + \cdots$$

$$p_2(s) = a_0^+ + a_1^+ s + a_2^- s^2 + a_3^- s^3 + a_4^+ s^4 + \cdots$$

$$p_3(s) = a_0^+ + a_1^- s + a_2^- s^2 + a_3^+ s^3 + a_4^+ s^4 + \cdots \qquad (3.34c)$$

$$p_4(s) = a_0^+ + a_1^- s + a_2^- s^2 + a_3^+ s^3 + a_4^+ s^4 + \cdots \qquad (3.34d)$$

The interval polynomial is robustly stable if and only if these four polynomials are Hurwitz stable.

One of the main limitations of Kharitonov's theorem is the restriction that the polynomial family must have the structure of an interval polynomial. An important result which applies to a much more general class of polynomial families is the edge theorem. This theorem applies to a polynomial family consisting of a collection of polynomials of the form:

$$p(s) = a_0(q) + a_1(q)s + \cdots + a_n(q)s^n$$

(28)

Where$q = [q_1 \ \ldots\ldots\ldots.. q_n]; q_i^- \le q_i \le q_i^+, i = 0,1,2, \ldots\ldots, k$ (29)

and where the functions $a_0(.), \ldots.. a_n(.)$ are affine linear. In such a polynomial family, the polynomial coefficients are contained in a polytope.

Design Of Full Order State Observer Using Kharitnov Polynomial

Characteristic equation of full order state observer is given by

$$\det(sI - A + K_e C) = 0 \tag{30}$$

Considering

$$K_e = \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix}$$

As observer gain matrix, the characteristic equation can be written as

$$s^3 + (32.04K_2 + 0.01573K_3 + 18.21)s^2 + (-256.32K_1 + 583.6K_2 + 0.286K_3)s$$
$$+K_1 + 2.29K_2 + 17889K_3 - 446.8998 = 0 \tag{31}$$

By applying routh criteria for finding the range by Kharitnov technique, the following constraints are obtained

$$4.56K_2 + 17889K_3 > 446.8998$$
$$2K_1 - 0.02K_2 + 17889K_3 > 446.8998$$
$$32.04K_2 + 0.01573K_3 > -18.21 \tag{32}$$

Solving the characteristic equation considering the above constraints, the range of values for which error becomes approximately zero is

$$|K_1| \geq 1.3, |K_2| \geq 0.56\ , |K_3|\ \geq 0.025$$

Case Study with Pole Placement Technique

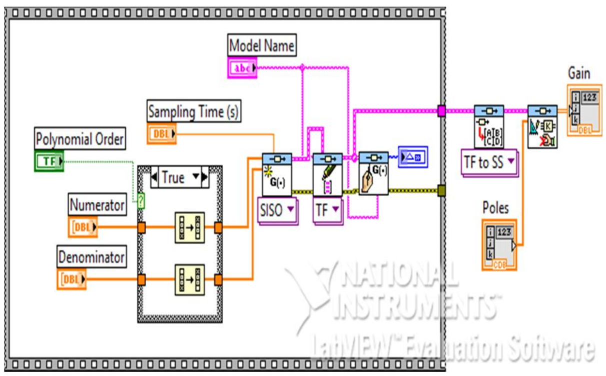

Fig 8: Block diagram for finding obsever gain matrix

Case study 1

Design of Full order state observer for RIP system with desired poles placed at s= -2, -2+j1, -2-j1

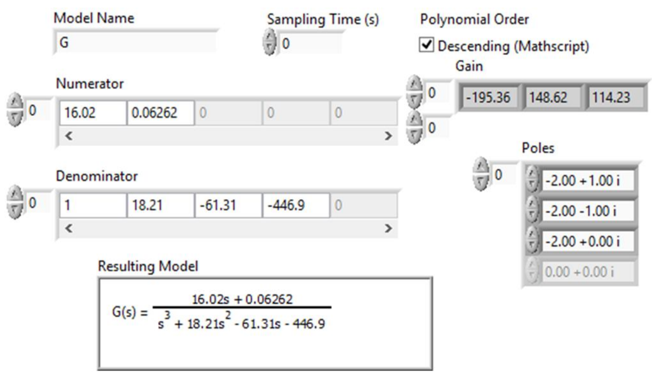

Fig 9: Front panel of LabVIEW

Case study 2
Design of Full order state observer for RIP system with desired poles placed at s= -1,-2,-10

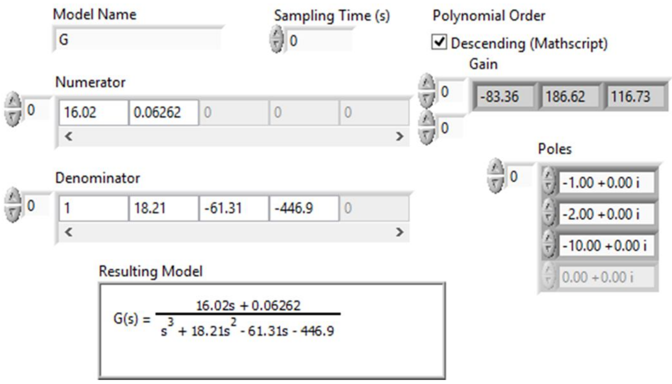

Fig 10: Front panel of LabVIEW

In both case stidies the observer gain matrix obtained is in adhrenece to the range of gain values.

Conclusion

In the research being carried out the system is taken as rotary inverted pendulum. It is a highly nonlinear and unstable system. Linearization is done with the help of state space equations. From the linearized model, the transfer function of the system with pendulum angle as output and input voltage to the motor is taken. Then from the characteristic polynomial that is the Kharitnov polynomial is formed with state feedback observer transfer function. By giving different ranges for state feedback observer gain matrix**,** the range which stabilizes the system is found ie all closed loop poles lies on the left side of the s plane in that range.

The future scope of the project are i) aircraft control**-** as a counter balancing system in the case of an aircraft ii) ship control- the centre of gravity remains same for the research proposed. So if there are different weights on both sides of the ship, then also ship can be stabilized on its centre of gravity by embedding this system and iii) automobile control.**-** During the event of different weights on both sides of the automobile, on curve of a road the automobile can stabilized on its centre of gravity.

References

[1] "Classical dual inverted pendulum control", Kent H Lundberg &James K Roberge, presented at 2003 IEEE conference on decision and control.
[2] "Stability Analysis and Design of PI ControllerUsing Kharitnov Polynomial for Rotary Inverted Pendulum", Jim George, Bipin Krishana, Shreesha C, V I George, Mukund Kumar Menon, Sensors& Transducers, Vol. 138, Issue 3, March 2012,pp:104-113.
[3] "Determination of the region of the stabilizing controller parameters for polytopic polynomials", Mukund Kumar Menon,I Thirunavakkarasu, V I George, Sensors and Transducers Journal,Vol.119, Issue8, August 2010,pp.174-181.
[4] "Minimum time swing up and stabilization of rotary inverted pendulum using pulse step control", P.Melba Mary and S.Marimuthu, Iranian Journal of fuzzy systems Vol. 6, No. 3(2009) pp.1-15.
[5] "Design,Build and Control of a Single/Double Rotational Inverted Pendulum", Final report, James Driver and Dylan Thorpe, School of Mechanical Engineering, University of Adelaide, Australia, October 2004.
[6] "Sliding mode control of suspended pendulum", Hakan Kizmaz & Sadettin Aksoy, Modern Electric power system conference 2010.
[7] "Stabilising controller design for rotary inverted pendulum" Bipin Krishna, I T Arasu, Dr. V I George, National conference on communication networks and sensor technology 2011 proceedings page no: 285-288
[8] "Design and Simulation of Dead Beat Model and Minimum-Order State Observer for Rotary Inverted Pendulum" Proceedings of 'International Conference on Science & Technology:Applications in Industry & Education (2010)' (ICSTIE-2010), Universiti Teknology MARA Pulau Pinang, Malaysia, Dec-2010
[9] K Ogatta, "Modern Control Theory", PHI publications, 2010
[10] ni.com/LabVIEW/ control design toolbox